Proof of DeMoivre's Theorem

We want to prove that $\left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta$.

We shall prove this using Mathematical Induction. To do this, we follow these steps:

Step (i): Show that the statement is true for $n = 1$.

Step (ii): For any natural number k if the statement is true for $n = k$, then logically it is true for $n = k + 1$.

If steps (i) and (ii) are true, then we have that the statement is true for all natural numbers n.

Notice that we must assume that the statement holds for $n = k$ to show that it logically follows that the statement is also true for the case when $n = k + 1$. This assumption is called the inductive hypothesis.

If we plug in $n = 1$, we see that both sides are equal, so the statement is true. Now, we assume that the statement is true for $n = k$, that is $\left(\cos\theta + i\sin\theta\right)^k = \cos k\theta + i\sin k\theta$. We need to show that $\left(\cos\theta + i\sin\theta\right)^{k+1} = \cos\left[\left(k+1\right)\theta\right] + i\sin\left[\left(k+1\right)\theta\right]$ follows logically.

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\begin{aligned}
\left(\cos\theta + i\sin\theta\right)^{k+1} &= \left(\cos\theta + i\sin\theta\right)^k \left(\cos\theta + i\sin\theta\right) \\
&= \left(\cos k\theta + i\sin k\theta\right)\left(\cos\theta + i\sin\theta\right) && \text{(By hypothesis)} \\
&= \left(\cos k\theta\right)\left(\cos\theta\right) + \left(\cos k\theta\right)\left(i\sin\theta\right) \\
&\quad + \left(i\sin k\theta\right)\left(\cos\theta\right) + \left(i\sin k\theta\right)\left(i\sin\theta\right) \\
&= \left[\left(\cos k\theta\right)\left(\cos\theta\right) - \left(\sin k\theta\right)\left(\sin\theta\right)\right] \\
&\quad + i\left[\left(\sin k\theta\right)\left(\cos\theta\right) + \left(\cos k\theta\right)\left(\sin\theta\right)\right] \\
&= \cos\left[\left(k+1\right)\theta\right] + i\sin\left[\left(k+1\right)\theta\right] && \text{(By add/sub formulas)}
\end{aligned}
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Since we have shown that the statement is true for the case of $n = 1$ and also that $\left(\cos\theta + i\sin\theta\right)^k = \cos k\theta + i\sin k\theta \Rightarrow \left(\cos\theta + i\sin\theta\right)^{k+1} = \cos\left[\left(k+1\right)\theta\right] + i\sin\left[\left(k+1\right)\theta\right]$ we can conclude that the formula holds for all n, and thus we are done.