chapter 7

Algebra 3

Section 7.12 Proofs by induction

To prove a statement true by induction, we follow clearly-defined steps.

(i) The statement is proven true for some fixed value, usually $n = 1$ or $n = 2$.
(ii) The statement is then assumed true for values up to $n = k$.
(iii) Based on this assumption, we must show that the statement is true for $n = k + 1$.
(iv) In conclusion, a "rolling proof" is formed:
   a. Since it was true for $n = 1$,
   b. it is now true for $n = 1 + 1 = 2$.
   c. Since it is true for $n = 2$, it is true for $n = 2 + 1 = 3$, etc.
   d. It is therefore true for all values of n.

Results involving **series of numbers**

Example 1

Prove that **for all values of** n, $1 + 2 + 3 + 4 + \ldots . n = \frac{n}{2}(n+1)$.

① Show true for n=1

Is $1 = \frac{1}{2}(1+1)$?
$1 = 1$ yes.

② Assume true for n=k

Assume: $1+2+\ldots+k = \frac{k}{2}(k+1)$

③ To prove true for $n = k+1$

Is $\underline{1+2+\ldots+k}+(k+1) \stackrel{?}{=} \frac{k+1}{2}(k+1+1)$

$\frac{k}{2}(k+1) + k+1 \stackrel{?}{=} \frac{(k+1)}{2}(k+2)$

$\frac{k(k+1)+2k+2}{2} \stackrel{?}{=}$ RHS

$\frac{k^2+k+2k+2}{2} \stackrel{?}{=}$ RHS

$\frac{k^2+3k+2}{2} \stackrel{?}{=}$ RHS

$\frac{(k+1)(k+2)}{2} \stackrel{?}{=}$ RHS ✓ YES

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④ State finding

Its true for $n=1$, $n=k$ and $n=k+1$
$\Rightarrow$ true for all $n \in \mathbb{N}$

Divisibility proofs

Example 4

Prove that for all $n \in N$, 3 is a factor of $4^n - 1$.

① Show true for n=1

Is $4^1 - 1$ divisible by 3
3 divisible by 3 YES

② Assume true for n=k

Assume $4^k - 1$ is divisible by 3

③ To prove true for $n = k+1$

To prove $4^{k+1} - 1$ is divisible by 3

$4^{k+1} - 1 = (4)4^k - 1$

$f(k+1) - f(k)$ $= (4)4^k - \cancel{1}$ $\mp 4^k \pm \cancel{1}$ $= (3)4^k$

which is divisible by 3
$\Rightarrow 4^{k+1}$ is divisible by 3

$f(k+1) - f(k)$ = number that is divisible...

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④ State finding

Its true for $n=1$, $n=k$ and $n=k+1$
$\Rightarrow$ true for all $n \in \mathbb{N}$

Example 5

Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all $n \in N$.

① Show true for n=1

Is $8^1 - 7(1) + 6$ divisible by 7
$= 14 - 7 = 7 \Rightarrow$ yes

② Assume true for n=k

Assume $8^k - 7k + 6$ is divisible by 7

③ To prove true for n= k+1

Is $8^{k+1} - 7(k+1) + 6$ divisible by 7 ?

$$f(k+1) = (8)8^k - \cancel{7k} - 7 + \cancel{6}$$
$$-f(k) \quad \mp 8^k \pm \cancel{7k} \mp \cancel{6}$$

$$f(k+1) - f(k) = 7(8^k) - 7$$
$$= 7(8^k - 1)$$

which is divisible by 7

$\Rightarrow f(k+1)$ is divisible by 7

If $f(k+1) - f(k)$ is divisible by the same no. that is a factor of $f(k)$
$\Rightarrow$ its a factor of $f(k+1)$

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④ State finding

Its true for n=1, n=k and n=k+1
$\Rightarrow$ true for all $n \in N$